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TITLE OF PAPER IN ENGLISH

Abstract

In the present paper, we study strongly minimal partial orderings in the signature containing only the symbol of binary relation of partial order. We use for partial orderings such characteristics as the height of a structure that is the supremum of lengths of ordered chains, and the width of a structure that is the supremum of lengths of antichains, where an antichain is a set of pairwise incomparable elements. The main result of the paper is a criterion for strong minimality of an infinite partial ordering of height two having an infinite non-trivial width.

Key words: strongly minimal structure, partial ordering, connected component, maximal element, minimal element.

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Introduction

Recall [1] that an infinite structure M is said to be *minimal* if for any formula $\varphi(x)$ of the language of M , with parameters from M , either $\varphi(M)$ or $\neg\varphi(M)$ is finite. A theory T without finite models is said to be *strongly minimal* if any model M of T is minimal. Models of a strongly minimal theory are said to be strongly minimal, too.

Recall that a partial order on a set is a binary relation $<$ satisfying:

Asymmetry $\forall x \forall y [x < y \rightarrow \neg(y < x)]$;

Transitivity $\forall x \forall y \forall z [(x < y \wedge y < z) \rightarrow x < z]$.

A set equipped with a partial order on it is a *partially ordered set* or *partial ordering*.

Let $M = \langle M, < \rangle$ be an infinite partial ordering. Assuming that M is strongly minimal we have only finite $\leq$ -chains and the lengths of these chains are bounded, since unbounded lengths imply existence of structures N that are elementarily equivalent to M with infinite chains $\{a_i \mid i \in \mathbb{Z}\}$, $a_i < a_j$ for $i < j$, that violate the strong minimality by any formula $x < a_i$. Thus, the height $h(M)$, that is the supremum of lengths of $\leq$ -chains in M , must be finite for a strongly minimal structure M . Therefore, further we consider here only infinite partial orderings with finite $\leq$ -chains.

Also, since M is infinite, it has an infinite width $w(M)$ that is the supremum of cardinalities of $\leq$ -antichains.

For any infinite partial ordering with finite $\leq$ -chains $M = \langle M, < \rangle$ we can consider the corresponding graph structure $M_R = \langle M, R^2 \rangle$, where $R(a, b)$ iff either a is an immediate predecessor of b or a is an immediate successor of b with regard to the partial order $<$ for any $a, b \in M$. We say a subset A of a partial ordering $M = \langle M, < \rangle$ is a connected component if A is a connected component in $M_R = \langle M, R^2 \rangle$.

We say a connected component is trivial if it is a chain. We say a trivial connected component is an n -component if it is a chain of length n , where $1 \leq n < \omega$. We say a 1-component is a singleton.

The following theorem describes strongly minimal partial orderings with finite non-trivial width:

Theorem 1 [2, 3] *An infinite partial ordering $M = \langle M, < \rangle$ with a finite non-trivial width is strongly minimal iff M has infinitely many singletons and additionally can have only finitely many finite connected components which are not singletons.*

In the recent paper [3], strongly minimal partial orderings having a finite non-trivial width were completely described. Here we study strongly minimal partial orderings having an infinite non-trivial width. In the present time, the class of strongly minimal theories is an object of active investigations. In [4], rank properties for families of strongly minimal theories were studied. In [5], some interesting results on strongly minimal structures were obtained that lead to a finer classification of strongly minimal structures with flat geometry.

Materials and Methods

Here we study partial orderings in the signature containing only the binary relation of partial order. Methods using for studying are classical methods of Model Theory, in particular, methods of studying strongly minimal structures.

Results and Discussion

Recall that an element a of a partial ordering M is said to be minimal if there is no element in M that is less than a . Also, an element a of a partial ordering M is said to be maximal if there is no element in M that is greater than a . Observe that any singleton of a partial ordering is both maximal and minimal element of the ordering. Also, any maximal (minimal) element of a non-trivial connected component of a partial ordering is not minimal (maximal).

Example 2 Let $M = \langle M, < \rangle$ be a partial ordering consisting of exactly one infinite non-trivial connected component of height two having exactly one maximal element. Then, obviously, M is a strongly minimal structure having an infinite non-trivial width.

We say that an element a of a partial ordering M has up-degree (down-degree) m for some $m \in \omega$ if there exist exactly m elements of M being an immediate successor (predecessor) of a . If there are infinitely many elements of M being an immediate successor (predecessor) of a then we say a has infinite up-degree (down-degree). Obviously, a has both up-degree 0 and down-degree 0 iff a is a singleton.

Theorem 3 *Let $M = \langle M, < \rangle$ be an infinite partial ordering of height two having an infinite non-trivial width. Then M is strongly minimal iff the following holds:*

- (1) M contains exactly one non-trivial connected component of height two having both an infinite set of maximal (minimal) elements and a non-empty finite set of minimal (maximal) elements;
- (2) If M contains exactly m minimal (maximal) elements of infinite up-degree (down-degree) for some $1 \leq m < \omega$ then almost all maximal (minimal) elements have down-degree (up-degree) m ;
- (3) M contains only finitely many finite connected components of height at most two.

Proof of Theorem 3. ($\Rightarrow$) Let M be a strongly minimal structure. Since M has an infinite non-trivial width, we have that M contains infinitely many non-trivial connected components or at least one infinite non-trivial connected component. If M contains infinitely many non-trivial connected components then we obtain that M contains both infinitely many minimal elements and infinitely many maximal elements. This contradicts the strong minimality of M . Consequently, M contains only finitely many non-trivial connected components... $\square$

Recall that a first order theory is said to be totally categorical if it has exactly one model in each infinite power.

Corollary 4 *Any strongly minimal partial ordering of height two having an infinite non-trivial width is totally categorical.*

Conclusion

Thus, we obtained a complete description of strongly minimal partial orderings of height two having an infinite non-trivial width. As a corollary, we have that such strongly minimal partial orderings are totally categorical.

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НАЗВАНИЕ СТАТЬИ НА РУССКОМ ЯЗЫКЕ

Аннотация

В настоящей статье мы исследуем сильно минимальные частичные порядки в сигнатуре, содержащей только символ бинарного отношения, выражающего частичный порядок. Мы используем для частичных порядков такие характеристики, как высота структуры, означающая супремум длин упорядоченных цепей, и ширина структуры, означающая супремум длин антицепей, где антицепь – это множество попарно несравнимых элементов. Основной результат статьи – критерий сильной минимальности бесконечного частичного порядка высоты два, имеющие бесконечную нетривиальную ширину.

Ключевые слова: сильно минимальная структура, частичный порядок, связная компонента, максимальный элемент, минимальный элемент.

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НАЗВАНИЕ СТАТЬИ НА КАЗАХСКОМ ЯЗЫКЕ

Аңдатпа

Бұл зерттеуде біз жартылай ретті білдіретін бинарлық қатынастың таңбасымен ғана анықталатын қол таңбалардағы күшті минималды жартылай реттіліктерді талдаймыз. Жартылай реттіліктер үшін құрылымның биіктігі (реттелген тізбектердің ұзындықтарының жоғарғы шегі) және құрылымның ені (антитізбектердің ұзындықтарының жоғарғы шегі) сияқты сипаттамалар қолданылады. Мұнда антитізбек деп өзара салыстырылмайтын элементтер жиынтығы түсіндіріледі. Осы жұмыста біз шексіз тривиалды емес ені бар күшті минималды жартылай реттіліктерді қарастырамыз. Мақаланың негізгі нәтижесі – шексіз тривиалды емес ені бар және биіктігі екіге тең шексіз жартылай реттіліктердің күшті минималдылығы үшін қажетті және жеткілікті шарттың анықталуы.

Тірек сөздер: күшті минималды құрылым, жартылай реттілік, жалғанған компонент, максималды элемент, минималды элемент.